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The weak repulsion property

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Abstract

In 1926 M. Lavrentiev [M. Lavrentiev, Sur quelques problèmes du calcul des variations, An. Mat. Pura Appl. 4 (1926) 7–28] proposed an example of a variational problem whose infimum over the Sobolev space $\mathbf{W}^{1,p}$, for some values of $p \geqslant 1$, is strictly lower than the infimum over $\mathbf{W}^{1,\infty}$. This energy gap is known since then as the Lavrentiev phenomenon.

The aim of this paper is to provide a deeper insight into this phenomenon by shedding light on an unnoticed feature. Any energy that presents the Lavrentiev gap phenomenon is unbounded in any neighbourhood of any minimizer in $\mathbf{W}^{1,p}$.

We also show a finer result in case of regular minimizers and the repulsion property (observed by J. Ball and V. Mizel [J.M. Ball, V.J. Mizel, One-dimensional variational problems whose minimizers do not satisfy the Euler–Lagrange equation, Arch. Rational Mech. Anal. 90 (4) (1985) 325–388]) for any power $\alpha > 1$ of a Lagrangian that exhibits the Lavrentiev gap phenomenon.

Résumé

En 1926 M. Lavrentiev [M. Lavrentiev, Sur quelques problèmes du calcul des variations, An. Mat. Pura Appl. 4 (1926) 7–28] découvrit un exemple de problème variationnel dans lequel l'infimum sur l'espace de Sobolev $\mathbf{W}^{1,p}$, pour des $p \geqslant 1$, est strictement inférieur à l'infimum sur l'espace $\mathbf{W}^{1,\infty}$. Ce saut d'énergie est connu sous le nom de phénomène de Lavrentiev.

Dans cet article on présente une nouvelle caractéristique liée à ce phénomène qui permet de mieux le comprendre. Toutes les énergies qui présentent un saut de Lavrentiev sont non bornées dans tous les voisinages de chaque minimum en $\mathbf{W}^{1,p}$.

Finalement nous donnons un résultat plus précis dans le cas où les minima sont réguliers, et nous démontrons la propriété de répulsion (observée par J. Ball et V. Mizel [J.M. Ball, V.J. Mizel, One-dimensional variational problems whose minimizers do not satisfy the Euler–Lagrange equation, Arch. Rational Mech. Anal. 90 (4) (1985) 325–388]) pour toutes les puissances $\alpha > 1$ d'une lagrangienne présentant le phénomène de Lavrentiev.

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1. Introduction

In 1926 M. Lavrentiev [7] published an example of an action functional:

$$\mathcal{I}(u) := \int_a^b L(t, u, \dot{u})\, \mathrm{d}t,$$

whose infimum over the space $\mathbf{W}^{1,p}(a, b)$, for some $p \geqslant 1$, is strictly lower than the infimum over the space $\mathbf{W}^{1,\infty}(a, b)$, with fixed boundary conditions. This energy gap is known as the Lavrentiev phenomenon since then.

It is a manifestation of the high sensibility of the variational formulation upon the set of admissible minimizers (considering that $\mathbf{W}^{1,\infty}(a, b)$ is dense in $\mathbf{W}^{1,p}(a, b)$). Notice that an unpleasant drawback of this phenomenon is the impossibility of computing the minimizer and the minimum of the energy by a standard finite-element scheme.

One simple example exhibiting such energy gap is given by the Manià's action [8]: for any $\alpha \geqslant 9/2$, the functional,

$$\mathcal{I}_\alpha(u) := \int_0^1 (u^3 - t)^2 |\dot{u}|^\alpha\, \mathrm{d}t,$$

with boundary conditions $u(0) = 0$, $u(1) = 1$, is such that

$$\inf\{\mathcal{I}_\alpha : \mathbf{W}_t^{1,p}(0, 1)\} < \inf\{\mathcal{I}_\alpha \colon \mathbf{W}_t^{1,\infty}(0, 1)\}$$

(where $\mathbf{W}_t^{1,p}(0, 1)$ denotes the space $t + \mathbf{W}_0^{1,p}(0, 1)$), for p in $[1, 3/2)$. More examples verifying this phenomenon have been given by several authors: see [5,9] and references therein.

A further singular phenomenon can be observed when α is strictly greater than $9/2$. Any sequence $\{u_n\}$ in $\mathbf{W}_t^{1,\infty}(0, 1)$ which converges almost everywhere in $(0, 1)$ to the minimizer of $\mathcal{I}_\alpha$ is such that

$$\mathcal{I}_\alpha(u_n) \to +\infty.$$

That is called repulsion property [2]. It was observed by J. Ball and V. Mizel in 1985 [3] for a energy slightly different from $\mathcal{I}_\alpha$. Fairly surprisingly, the closer we approximate the minimizer the farther we escape from the minimum value of the energy.

The aim of this paper is to investigate the relations between Lavrentiev phenomenon and a "weak" repulsion property.

Our main result states that, if

$$\mathcal{I}(u) := \int_\Omega L(x, u, \nabla u)\, \mathrm{d}x$$

(where Ω is an open bounded set in $\mathbb{R}^N$, u is defined on Ω with values in $\mathbb{R}^m$ and $L(x, u, w)$ is a Carathéodory function) presents the Lavrentiev phenomenon, then for any minimizer $\bar{u}$ for $\mathcal{I}$ in $\mathbf{W}_\psi^{1,p}(\Omega)$, there exists a sequence $\{\bar{u}_n\}$ in $\mathbf{W}_\psi^{1,\infty}(\Omega)$ which converges to $\bar{u}$ in $\mathbf{W}_\psi^{1,p}(\Omega)$, such that

$$\mathcal{I}(\bar{u}_n) \to +\infty$$

(where ψ is a fixed function in $\mathbf{W}^{1,\infty}(\Omega)$ and $\mathbf{W}_\psi^{1,p}(\Omega)$ denotes the space $\psi + \mathbf{W}_0^{1,p}(\Omega)$).

In other words, in a variational model we observe the "weak" repulsion property as soon as we observe the Lavrentiev phenomenon (our main result is true under practically no assumptions). In dimension one and for a special but rather large class of Lagrangians, in [6] a similar result than the one we present here has been shown. Namely, the Lavrentiev phenomenon implies that any minimizer $\bar{u}$ in $\mathbf{W}^{1,1}$ is the limit of a sequence $\{\tilde{u}_n\}$ in $\mathbf{W}^{1,1}$ such that $\mathcal{I}(\tilde{u}_n)$ is identically equal to $+\infty$ (by using the Fatou's lemma, from that result it is easy to deduce the existence of a sequence $\{\bar{u}_n\}$ in $\mathbf{W}^{1,\infty}$ converging to $\bar{u}$ in $\mathbf{W}^{1,1}$ such that $\mathcal{I}(\bar{u}_n) \to +\infty$).

We would like to point out that the repulsion property is a property related to any sequence that converges to a minimizer whereas our "weak" repulsion property is a property related to (at least) one specific sequence among those that converge to a minimizer. On the contrary, our result would have been false since the Lavrentiev phenomenon does not imply the repulsion property (as the Manià's energy $\mathcal{I}_\alpha$, for $\alpha = 9/2$, shows). Our main theorem is therefore optimal in that sense.

The paper is organized as follows. In Section 2 we give the proofs of the afore mentioned claims on the Manià's functional for reader convenience.

In Section 3 we present our main result and the case of continuous and $\mathbf{L}^q$, $q \geqslant 1$, minimizers.

In Section 4 we show the repulsion property for a modified Lagrangian. More precisely, we prove that, if an action with Lagrangian L exhibits the Lavrentiev phenomenon, then the action with the modified Lagrangian $\phi \circ L$ manifests the repulsion property, for any function $\phi : \mathbb{R} \to \mathbb{R}$ with super-linear growth.

2. The Manià's example

For reader convenience, we present briefly an overview of the Manià's example. No original results are contained in this section (though the Manià's functional proposed here differs slightly from the usual one). An alternative source for this example can be found in [5], for instance.

In Proposition 1, we show that, for $\alpha \geqslant 9/2$, the functional $\mathcal{I}_\alpha$ exhibits the Lavrentiev gap phenomenon and, in Proposition 2, that $\mathcal{I}_\alpha$ presents the repulsion property, for $\alpha > 9/2$. We stress that the limit case $\alpha = 9/2$ the repulsion property, does not occur. It is explained at the end of this section.

Proposition 1. *For any $\alpha \geqslant 9/2$, the functional,*

$$\mathcal{I}_\alpha(u) := \int_0^1 (u^3 - t)^2 |\dot{u}|^\alpha \, \mathrm{d}t,$$

with boundary conditions $u(0) = 0$, $u(1) = 1$, presents the Lavrentiev phenomenon, i.e.

$$\inf\{\mathcal{I}_\alpha \colon \mathbf{W}_t^{1,p}(0,1)\} < \inf\{\mathcal{I}_\alpha \colon \mathbf{W}_t^{1,\infty}(0,1)\},$$

for p in $[1, 3/2)$.

Proof. The Lagrangian associated to $\mathcal{I}_\alpha$ has non-negative values. Since $\mathcal{I}_\alpha$ evaluated in $\bar{u}(t) = \sqrt[3]{t}$ is identically zero, we obtain that $\bar{u}$ is a minimizer of $\mathcal{I}_\alpha$ in $\mathbf{W}_t^{1,p}(0,1)$, for p in $[1, 3/2)$.

Let u be any function in $\mathbf{W}_t^{1,\infty}(0,1)$. Since $\dot{u}$ is essentially bounded, there exists a real number a in $(0,1)$ such that $u(t) < \sqrt[3]{t}/2$, for any t in $[0,a]$, and $u(a) = \sqrt[3]{a}/2$. Hence,

$$[u^3 - t]^2 \dot{u}^\alpha \geqslant \left[\frac{t}{2^3} - t\right]^2 \dot{u}^\alpha = \frac{7^2}{8^2} t^2 \dot{u}^\alpha,$$

for any t in $[0,a]$.

For $\alpha > 3$, by the Hölder inequality, we obtain:

$$\begin{aligned}\frac{\sqrt[3]{a}}{2} = \int_0^a t^{-2/\alpha} t^{2/\alpha} u' \, \mathrm{d}t &\leqslant \left(\int_0^a t^{-2/(\alpha-1)} \, \mathrm{d}t\right)^{(\alpha-1)/\alpha} \left(\int_0^a t^2 \dot{u}^\alpha \, \mathrm{d}t\right)^{1/\alpha} \\ &= \left(\frac{\alpha-1}{\alpha-3}\right)^{(\alpha-1)/\alpha} a^{(\alpha-3)/\alpha} \left(\int_0^a t^2 \dot{u}^\alpha \, \mathrm{d}t\right)^{1/\alpha}.\end{aligned}$$

We conclude that, for any u in $\mathbf{W}_t^{1,\infty}(0,1)$,

$$\mathcal{I}_\alpha(u) \geqslant \int_0^a (u^3 - t)^2 \dot{u}^\alpha \, \mathrm{d}t \geqslant a^{3-2\alpha/3} \frac{7^2}{8^2} \left(\frac{\alpha-3}{\alpha-1}\right)^{(\alpha-1)} \geqslant \frac{7^2}{8^2} \left(\frac{\alpha-3}{\alpha-1}\right)^{\alpha-1} > 0, \tag{1}$$

for $\alpha \geqslant 9/2$. $\square$

Proposition 2. *If $\alpha > 9/2$, then*

$$\mathcal{I}_\alpha(u_n) \to +\infty,$$

for any sequence $\{u_n\}$ in $\mathbf{W}_t^{1,\infty}(0,1)$ converging a.e. to $\bar{u}$ in $(0,1)$.

Proof. For any natural number n, let a_n in $(0,1)$ be such that $u_n(t) < \sqrt[3]{t}/2$, for any t in $(0, a_n)$, and $u(a_n) = \sqrt[3]{a_n}/2$. By the convergence of u_n to $\bar{u}$, we have that a_n tends to 0.

Using the inequality (1), since p is strictly greater then $9/2$, we conclude that

$$\mathcal{I}_\alpha(u_n) \geqslant a_n^{3-2\alpha/3} \frac{7^2}{8^2}\left(\frac{\alpha-3}{\alpha-1}\right)^{\alpha-1} \to +\infty,$$

as n goes to ∞. □

Notice that for $\alpha = 9/2$, Proposition 2 is not true. Indeed, consider the sequence in $\mathbf{W}_t^{1,\infty}(0,1)$ given by:

$$u_n(t) := \begin{cases} n^{2/3}t, & t \in [0, 1/n], \\ \sqrt[3]{t}, & t \in (1/n, 1], \end{cases}$$

which converges a.e. to $\bar{u}$, and also in $\mathbf{W}_t^{1,1}(0,1)$ since

$$\int_0^1 |\dot{u}_n - \dot{\bar{u}}| \,\mathrm{d}t = \int_0^{1/n} |n^{2/3} - 3^{-1}t^{-2/3}| \,\mathrm{d}t = \int_0^{1/(3^{3/2}n)} (3^{-1}t^{-2/3} - n^{2/3}) \,\mathrm{d}t + \int_{1/(3^{3/2}n)}^{1/n} (n^{2/3} - 3^{-1}t^{-2/3}) \,\mathrm{d}t$$
$$= \left(1 - \frac{4}{3^{3/2}}\right) n^{-1/3} \to 0.$$

Let us verify that $\mathcal{I}_\alpha(u_n)$ is bounded for any $\alpha \leqslant 9/2$. In fact, for any n,

$$\mathcal{I}_\alpha(u_n) = \int_0^{1/n} (n^2t^3 - t)^2 n^{2\alpha/3} \,\mathrm{d}t = n^{2\alpha/3} \int_0^{1/n} (n^4t^6 + t^2 + 2n^2t^4) \,\mathrm{d}t = n^{2\alpha/3-3}\left(\frac{1}{7} + \frac{1}{3} + \frac{2}{5}\right).$$

Therefore, for $\alpha = 9/2$, $\mathcal{I}_{9/2}(u_n)$ is bounded by the constant $1/7 + 1/3 + 2/5$.

Notice that the last equality also proves that, when α belongs to $(0, 9/2)$, $\mathcal{I}_\alpha$ does not manifest the Lavrentiev gap since $\mathcal{I}_\alpha(u_n)$ converges to $0 = \mathcal{I}_\alpha(\bar{u})$.

3. The weak repulsion property

The framework we shall work with in the sequel is the following.

Let Ω be an open bounded set in $\mathbb{R}^N$. We denote by $\mathbf{W}^{1,p}(\Omega)$, $p \geqslant 1$, the Sobolev space of vector-valued functions $u : \Omega \to \mathbb{R}^m$. Given ψ in $\mathbf{W}^{1,\infty}(\Omega)$, $\mathbf{W}_\psi^{1,p}(\Omega)$ is as usual the space

$$\psi + \mathbf{W}_0^{1,p}(\Omega).$$

We assume that the Lagrangian $L(x,u,w) : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times N} \to \overline{\mathbb{R}}$ is a Carathéodory function, i.e. measurable with respect to x and continuous with respect to u and w, bounded from below by an integrable non-positive function $-\rho(x)$, uniformly in u, w.

We deal with an energy,

$$\mathcal{I}(u) := \int_\Omega L(x, u, \nabla u) \,\mathrm{d}x$$

that presents the Lavrentiev phenomenon, i.e. for some $p \geqslant 1$

$$\inf\{\mathcal{I}(u) \colon u \in \mathbf{W}_\psi^{1,p}(\Omega)\} < \inf\{\mathcal{I}(u) \colon u \in \mathbf{W}_\psi^{1,\infty}(\Omega)\}.$$

Our main result is stated right below.

Theorem 3. *Let $\bar{u}$ in $\mathbf{W}^{1,p}_{\psi}(\Omega)$ be a minimizer of $\mathcal{I}$.*

Then, there exists a sequence $\{\bar{u}_n\}$ in $\mathbf{W}^{1,\infty}_{\psi}(\Omega)$ which converges to $\bar{u}$ in $\mathbf{W}^{1,p}$ such that

$$\mathcal{I}(\bar{u}_n) \to +\infty.$$

Proof. We prove the theorem for $\psi = 0$. The general case can be reduced to that one replacing the Lagrangian L by:

$$L_{\psi}(x, u, w) := L\big(x, \psi(x) + u, \nabla\psi(x) + w\big).$$

We can also suppose $m = 1$ since the general case can be obtained by proceeding componentwise as follows.

Fix $\epsilon > 0$. Let $\{v_n\}$ be a sequence in $\mathbf{C}^{\infty}_c(\Omega)$ such that

$$\|v_n - \bar{u}\|_{\mathbf{W}^{1,p}(\Omega)} \leqslant \epsilon 2^{-n}$$

and that v_n and ∇v_n converge a.e. respectively to $\bar{u}$ and $\nabla\bar{u}$.

For any natural number n, consider the function $\mathcal{L}_n$ in $\mathbf{L}^1(\Omega)$ given by:

$$\mathcal{L}_n(x) := \max\big\{L\big(x, v_1(x), \nabla v_1(x)\big), \ldots, L\big(x, v_n(x), \nabla v_n(x)\big)\big\}.$$

Step 1. We claim that $\int_{\Omega}[\mathcal{L}_n]^+ \to +\infty$, where $[\mathcal{L}_n]^+$ denotes the positive part of $\mathcal{L}_n$.

Suppose that this is not true.

Notice that the sequence $\{\mathcal{L}_n\}$ is non-decreasing and $\mathcal{L}_n \geqslant L(x, v_n, \nabla v_n)$. Since by assumptions $L(x, v_n, \nabla v_n) \geqslant -\rho$, we have also the inequality:

$$\big|L(x, v_n, \nabla v_n)\big| \leqslant \max\big\{[\mathcal{L}_n]^+, \rho\big\}. \tag{2}$$

The monotonicity of the sequence $\{[\mathcal{L}_n]^+\}$ implies that, for any measurable set $E \subset \Omega$,

$$\int_E [\mathcal{L}_n]^+ \to l_E < +\infty.$$

By the Vitali–Hahn–Saks theorem [1], we conclude that $\{[\mathcal{L}_n]^+\}$ is equi-integrable and, by the estimate (2), $\{L(x, v_n, \nabla v_n)\}$ is equi-integrable too. Since $L(x, v_n, \nabla v_n)$ converges a.e. to $L(x, \bar{u}, \nabla\bar{u})$, we have that $L(x, v_n, \nabla v_n)$ converges weakly-$*$ in $\mathbf{L}^1(\Omega)$ to $L(x, \bar{u}, \nabla\bar{u})$. That implies

$$\lim_{n\to\infty} \mathcal{I}(v_n) = \mathcal{I}(\bar{u}),$$

which contradicts the presence of the Lavrentiev phenomenon.

Since $[\mathcal{L}_n]^- = \min\{L^-(x, v_1, \nabla v_1), \ldots, L^-(x, v_n, \nabla v_n)\} \leqslant \rho$, we have that

$$\int_{\Omega} \mathcal{L}_n \to +\infty.$$

Set $\mathcal{L}_0 := -\rho$ and let E_n be the measurable set defined by $\{x \in \Omega\colon\ \mathcal{L}_n(x) > \mathcal{L}_{n-1}(x)\}$. Observe that $\mathcal{L}_n = L(x, v_n, \nabla v_n)$ on E_n, whenever $|E_n| \neq 0$. By passing to a subsequence, we can suppose that $|E_n| \neq 0$, for any n.

For $k = 1, \ldots, n-1$, set $F^n_k := E_k \setminus (E_{k+1} \cup \cdots \cup E_n)$ and $F^n_n := E_n$. Notice that the F^n_k are pairwise disjoint sets. We have that

$$\mathcal{L}_n = \sum_{k=1}^{n} \mathcal{L}_k \chi_{F^n_k} = \sum_{k=1}^{n} L(x, v_k, \nabla v_k)\chi_{F^n_k}. \tag{3}$$

Step 2. We want to "regularize" the functions $\sum_{k=1}^{n} v_k \chi_{F^n_k}$.

Let $\mathcal{C}$ be the family of finite unions of disjoint open hypercubes of $\mathbb{R}^d$ with faces parallel to the coordinate hyperplanes. The sets in the family $\mathcal{C}$ have piecewise smooth boundary.

Since v_n has compact support contained in Ω, we can find a set Ω_n in the family $\mathcal{C}$ such that $\operatorname{supp}(v_n) \subset \Omega_n \subset \Omega$.

Fix n in $\mathbb{N}$.

For any $\delta > 0$ small enough, we define by induction on k, a set $G_k^n(\delta)$ in $\mathcal{C}$ such that $\bigcup_{i=k}^n F_i^n \subset G_k^n(\delta)$ except for a set of zero measure, $G_k^n(\delta) \subset G_{k-1}^n(\delta) \subset \Omega_n$ (or $G_k^n(\delta) \subset \Omega_n$, in case $k = 1$) and

$$\left| G_k^n(\delta) \setminus \left(\bigcup_{i=k}^n F_i^n \right) \right| \leqslant \delta. \tag{4}$$

Let us denote by $C(x_k^n(i); l_k^n(i))$ the hypercubes in the family $\mathcal{C}$ with centre in $x_k^n(i)$ and sides of length $l_k^n(i)$ such that $G_k^n(\delta) = \bigcup_{i=1}^{m_k^n} C(x_k^n(i); l_k^n(i))$.

Since $G_k^n(\delta)$ is an open set contained in Ω_n, for any η in $(0, \epsilon)$ small enough, we can define the following set:

$$\mathcal{G}_k^n(\eta) := \bigcup_{i=1}^{m_k^n} C\big(x_k^n(i); l_k^n(i) + \eta\big) \subset \Omega_n.$$

For any k, there exists a function $\rho_k^n(\delta, \eta)$ in $\mathbf{W}^{1,\infty}(\mathbb{R}^n)$ with values in [0, 1] such that

- $\rho_k^n(\delta, \eta)$ is identically 0 on $\mathbb{R}^n \setminus \mathcal{G}_k^n(\eta)$,
- $\rho_k^n(\delta, \eta)$ is identically 1 on $G_k^n(\delta)$,
- $\nabla\rho_k^n(\delta, \eta)$ is normal a.e. to the boundary of $G_k^n(\delta)$,
- $|\nabla\rho_k^n(\delta, \eta)|$ is identically equal to $2\eta^{-1}$ and, therefore, $\|\nabla\rho_k^n(\delta, \eta)\|_{\mathbf{L}^1(\mathbb{R}^n)}$ is bounded by a constant independent on k and n.

We can therefore define a function $u_n(\delta, \eta)$ in $\mathbf{W}_0^{1,\infty}(\Omega)$ by:

$$u_n(\delta, \eta) := v_1\rho_1^n(\delta, \eta) + \sum_{k=2}^n (v_k - v_{k-1})\rho_k^n(\delta, \eta).$$

It follows directly from the definition that

$$u_n(\delta, \eta) = v_k, \quad \nabla u_n(\delta, \eta) = \nabla v_k \quad \text{on } G_k^n(\delta) \setminus \left(\bigcup_{i=1,\ i\neq k}^n \mathcal{G}_i^n(\eta) \right).$$

Step 3. We give the sought sequence $\{\bar{u}_n\}$.

By the construction of $G_h^n(\delta)$ and $\mathcal{G}_h^n(\eta)$, we have that $u_n(\delta, \eta)$ and $\nabla u_n(\delta, \eta)$ converge a.e. respectively to v_k and ∇v_k on F_k^n, as δ and η tend to 0. Hence, the Fatou's lemma gives that $\int_\Omega \mathcal{L}_n \,\mathrm{d}x \leqslant \lim_{\eta,\delta\to 0} \mathcal{I}(u_n(\delta, \eta))$. We can therefore chose δ_n and η_n such that

$$\int_\Omega \mathcal{L}_n \,\mathrm{d}x - 1 \leqslant \mathcal{I}\big(u_n(\delta_n, \eta_n)\big). \tag{5}$$

Moreover, recalling that $\{\nabla v_n\}$ is convergent in $\mathbf{L}^p(\Omega)$ and that $\nabla\rho_k^n(\delta_n, \eta_n)$ is normal to $\partial G_k^n(\delta_n)$, δ_n and η_n can be chosen in such a way that we have as well the following inequalities:

$$\begin{cases} \|\nabla v_k - \nabla v_{k-1}\|_{\mathbf{L}^p(\partial G_k^n(\delta_n))} \leqslant \eta 2^{-k-1}, \\ \|(\nabla v_k - \nabla v_{k-1})|\nabla\rho_k^n(\delta_n, \eta_n)|^{1/p}\|_{\mathbf{L}^p(\Omega)} \leqslant \|\nabla v_k - \nabla v_{k-1}\|_{\mathbf{L}^p(\partial G_k^n(\delta_n))} + \eta 2^{-k-1} \end{cases} \tag{6}$$

(by passing to a subsequence of $\{\nabla v_n\}$ if needed). By the Poincaré inequality [4], there exists a constant $c_p > 0$ such that

$$\begin{aligned} \big\|(v_k - v_{k-1})\nabla\rho_k^n(\delta_n, \eta_n)\big\|_{\mathbf{L}^p(\Omega)} &\leqslant \|v_k - v_{k-1}\|_{\mathbf{L}^p(\mathcal{G}_k^n(\eta_n))} \big\|\nabla\rho_k^n(\delta_n, \eta_n)\big\|_{\mathbf{L}^\infty(\mathcal{G}_k^n(\eta_n)))} \\ &\leqslant c_p N^{1/(2p)} 2 \sum_{j=1}^{\overline{m}_k^n} \|\nabla v_k - \nabla v_{k-1}\|_{\mathbf{L}^p(C_j)} \end{aligned}$$

$$\leqslant c_p(2N^{1/2}\eta_n)^{1/p}2\sum_{j=1}^{\overline{m}_k^n}\big\|(\nabla v_k-\nabla v_{k-1})\big|\nabla\rho_k^n(\delta_n,\eta_n)\big|^{1/p}\big\|_{\mathbf{L}^p(C_j)}$$

$$\leqslant c_p(2N^{1/2}\eta_n)^{1/p}8\big\|(\nabla v_k-\nabla v_{k-1})\big|\nabla\rho_k^n(\delta_n,\eta_n)\big|^{1/p}\big\|_{\mathbf{L}^p(\Omega)}, \tag{7}$$

where $\{C_j\}_{j=1}^{\overline{m}_k^n}$ is the maximal subfamily of $\{C(x_k^n(i);l_k^n(i)+\eta_n)\}_{i=1}^{m_k^n}$ such that $\nabla\rho_k^n(\delta_n,\eta_n)$ is not identically zero on any hypercube C_j. Denote the functions $u_n(\delta_n,\eta_n)$ by u_n and by c the constant $c_p(2N^{1/2})^{1/p}8+1$.

By inequality (5), we have:

$$\mathcal{I}(u_n)\to+\infty.$$

Furthermore, one can verify that, for any n, $\|u_n-\bar{u}\|_{\mathbf{L}^p(\Omega)}\leqslant\epsilon$ and by (6) and (7),

$$\|\nabla u_n-\nabla\bar{u}\|_{\mathbf{L}^p(\Omega)}\leqslant 2\epsilon c.$$

(Recall that ϵ is an arbitrary positive constant given at the beginning of the proof.)

Set $\epsilon_n:=n^{-1}$. For any n, the sequence $\{u_n\}$ admits a subsequence $\{u_{k_n}\}$ such that $\|u_{k_n}-\bar{u}\|_{\mathbf{W}^{1,p}(\Omega)}\leqslant 3\epsilon_n c$. Denoting the functions u_{k_n} by $\bar{u}_n$, we have that the sequence $\{\bar{u}_n\}$ in $\mathbf{W}_0^{1,\infty}(\Omega)$ is such that

$$\|\bar{u}_n-\bar{u}\|_{\mathbf{W}^{1,p}(\Omega)}\to 0,$$

and

$$\mathcal{I}(\bar{u}_n)\to+\infty.$$

This concludes the proof. □

A local version of Theorem 3 can be briefly stated as follows:

$$\liminf_{u\xrightarrow{1,p}\bar{u},\ u\in\mathbf{W}_\psi^{1,\infty}(\Omega)}\mathcal{I}(u)\neq\mathcal{I}(\bar{u})\quad\Rightarrow\quad\limsup_{u\xrightarrow{1,p}\bar{u},\ u\in\mathbf{W}_\psi^{1,\infty}(\Omega)}\mathcal{I}(u)=+\infty.$$

We do not enter into details since it follows directly from the proof.

As a consequence of Theorem 3, we obtain the $\mathbf{W}^{1,p}$ local unboundedness of the energy $\mathcal{I}$. We have therefore the following corollary.

Corollary 4. *Let $\bar{u}$ in $\mathbf{W}_\psi^{1,p}(\Omega)$ be a minimizer of $\mathcal{I}$.*

If $\mathcal{I}$ is bounded in a neighbourhood of $\bar{u}$ in $\mathbf{W}^{1,p}$, then $\mathcal{I}$ does not manifest the Lavrentiev phenomenon.

In case we assume the continuity of the minimizer $\bar{u}$ in Theorem 3, we can improve the convergence of the repulsion sequence $\{\bar{u}_n\}$ to $\bar{u}$. More precisely:

Theorem 5. *Let $\bar{u}$ in $\mathbf{C}(\overline{\Omega})\cap\mathbf{W}_\psi^{1,p}(\Omega)$ be a minimizer of $\mathcal{I}$.*

Then, there exists a sequence $\{\bar{u}_n\}$ in $\mathbf{W}_\psi^{1,\infty}(\Omega)$ which converges to $\bar{u}$ in $\mathbf{C}\cap\mathbf{W}^{1,p}$ such that

$$\mathcal{I}(\bar{u}_n)\to+\infty.$$

Proof. The proof is slightly simpler than one of Theorem 3. We give a scheme of it using that theorem as reference.

We prove the theorem for $\psi=0$. The general case can be reduced to that one replacing the Lagrangian L by:

$$L_\psi(x,u,w):=L\big(x,\psi(x)+u,\nabla\psi(x)+w\big).$$

We can also suppose $m=1$ since the general case can be obtained by proceeding componentwise as follows.

Fix $\epsilon>0$. Let $\{v_n\}$ be a sequence in $\mathbf{C}_c^\infty(\Omega)$ such that

$$\|v_n-\bar{u}\|_{\mathbf{W}^{1,p}(\Omega)}\leqslant\epsilon 2^{-n},\qquad\|v_n-\bar{u}\|_{\mathbf{C}(\overline{\Omega})}\leqslant\epsilon 2^{-n}$$

and that v_n and ∇v_n converge a.e. respectively to $\bar{u}$ and $\nabla\bar{u}$.

For any natural number n, consider the function $\mathcal{L}_n$ in $\mathbf{L}^1(\Omega)$ given by:

$$\mathcal{L}_n(x) := \max\{L(x, v_1(x), \nabla v_1(x)), \ldots, L(x, v_n(x), \nabla v_n(x))\}.$$

Step 1 – *Step* 2. Those are exactly the same as in Theorem 3.

Step 3. We give the sought sequence $\{\bar{u}_n\}$.

By the construction of $G_h^n(\delta)$ and $\mathcal{G}_h^n(\epsilon)$, $u_n(\delta,\epsilon)$ and $\nabla u_n(\delta,\epsilon)$ converge a.e. respectively to v_k and ∇v_k on F_k^n, as δ and ϵ tend to 0. Hence, the Fatou's lemma gives that $\int_\Omega \mathcal{L}_n\,\mathrm{d}x \leqslant \lim_{\delta\to 0}\mathcal{I}(u_n(\delta))$. We can therefore chose δ_n and ϵ_n such that

$$\int_\Omega \mathcal{L}_n\,\mathrm{d}x - 1 \leqslant \mathcal{I}(u_n(\delta_n,\epsilon_n)). \tag{8}$$

Denoting the function $u_n(\delta_n,\epsilon_n)$ by u_n, by the inequality (8), we have that

$$\mathcal{I}(u_n) \to +\infty.$$

Furthermore, one can verify that, for any n, $\|u_n - \bar{u}\|_{\mathbf{C}(\overline{\Omega})} \leqslant \epsilon$ and, by the Hölder's inequality,

$$\|\nabla u_n - \nabla\bar{u}\|_{\mathbf{L}^p(\Omega)} \leqslant 2\epsilon.$$

(Recall that ϵ is an arbitrary positive constant given at the beginning of the proof.)

Set $\epsilon_n := n^{-1}$. For any n, the sequence $\{u_n\}$ admits a subsequence $\{u_{k_n}\}$ such that $\|u_{k_n} - \bar{u}\|_{\mathbf{C}(\overline{\Omega})} \leqslant \epsilon_n$ and $\|u_{k_n} - \bar{u}\|_{\mathbf{W}^{1,p}(\Omega)} \leqslant 3\epsilon_n$. Denoting the functions u_{k_n} by $\bar{u}_n$, we have that the sequence $\{\bar{u}_n\}$ in $\mathbf{W}_0^{1,\infty}(\Omega)$ is such that

$$\|\bar{u}_n - \bar{u}\|_{\mathbf{C}(\overline{\Omega})} \to 0, \qquad \|\bar{u}_n - \bar{u}\|_{\mathbf{W}^{1,p}(\Omega)} \to 0,$$

and

$$\mathcal{I}(\bar{u}_n) \to +\infty.$$

This concludes the proof. □

In case the minimizer $\bar{u}$ belongs to $\mathbf{L}^q(\Omega)$ with $q \geqslant 1$ (but $q \neq \infty$), we can be prove

Theorem 6. *Let $\bar{u}$ in $\mathbf{L}^q(\Omega) \cap \mathbf{W}_\psi^{1,p}(\Omega)$ be a minimizer of $\mathcal{I}$.*

Then, there exists a sequence $\{\bar{u}_n\}$ in $\mathbf{W}_\psi^{1,\infty}(\Omega)$ which converges to $\bar{u}$ in $\mathbf{L}^q \cap \mathbf{W}^{1,p}$ such that

$$\mathcal{I}(\bar{u}_n) \to +\infty.$$

Proof. The proof is exactly the same as the one of Theorem 3 except for the initial approximating sequence $\{v_n\} \subset \mathbf{C}_c^\infty(\Omega)$ which is chosen so that

$$\|v_n - \bar{u}\|_{\mathbf{W}^{1,p}(\Omega)} \leqslant \epsilon 2^{-n}, \qquad \|v_n - \bar{u}\|_{\mathbf{L}^q(\Omega)} \leqslant \epsilon 2^{-n},$$

and that v_n and ∇v_n converge a.e. respectively to $\bar{u}$ and $\nabla\bar{u}$. □

The reader might think that the weak repulsion property is satisfied by too many functionals. We would like to present below some simple examples which give proof of the contrary as we look for unbounded energies.

Notice that the deal of our main result is to provide a test to exclude the presence of the Lavrentiev phenomenon. As we deal with solutions that we already know belong to $\mathbf{W}_\psi^{1,\infty}(\Omega)$, it does not make any sense to apply Corollary 4 and Theorem 3 (that cannot even be applied in this case). Let us therefore restrict ourself to consider functionals with solutions in $\mathbf{W}_\psi^{1,1}(\Omega) \setminus \mathbf{W}_\psi^{1,\infty}(\Omega)$.

Let g be a function in $[\bigcap_{1\leqslant p<2}\mathbf{L}^p(0,1)] \setminus \mathbf{L}^2(0,1)$, $\int_0^1 g = 0$ (for instance, define $g(x) := x^{-1/2} - \int_0^1 x^{-1/2}\,\mathrm{d}x$) and consider the functional:

$$\mathcal{I}(u) := \int_0^1 [u'(x) - g(x)]^2\,\mathrm{d}x, \quad \text{for } u \text{ in } \mathbf{W}_0^{1,1}(0,1).$$

Denoting $\bar{u}(t) := \int_0^t g$, we have that

$$\mathcal{I}(u) \begin{cases} = 0, & u = \bar{u}, \\ > 0, & u \in \mathbf{W}_0^{1,1}(0,1) \setminus \mathbf{W}_0^{1,2}(0,1), \\ = +\infty, & u \in \mathbf{W}_0^{1,2}(0,1). \end{cases}$$

Analogously, let f be a function in $\mathbf{L}^1(0,1) \setminus [\bigcup_{p>1} \mathbf{L}^p(0,1)]$, $\int_0^1 f = 0$ (for instance, define $f(x) := [x \log^2(x/2)]^{-1} - \int_0^1 [x \log^2(x/2)]^{-1}\,\mathrm{d}x$) and consider, for $\alpha > 1$, the functional:

$$\mathcal{J}_\alpha(u) := \int_0^1 \left[u'(x) - f(x)\right]^\alpha \mathrm{d}x, \quad \text{for } u \text{ in } \mathbf{W}_0^{1,1}(0,1).$$

Denoting $\tilde{u}(t) := \int_0^t f$, we have that

$$\mathcal{J}_\alpha(u) \begin{cases} = 0, & u = \tilde{u}, \\ > 0, & u \in \mathbf{W}_0^{1,1}(0,1) \setminus \mathbf{W}_0^{1,\alpha}(0,1), \\ = +\infty, & u \in \mathbf{W}_0^{1,\alpha}(0,1). \end{cases}$$

The energies $\mathcal{I}$ and $\mathcal{J}_\alpha$ are indeed unbounded in any neighbourhood of $\bar{u}$ and $\tilde{u}$ in $\mathbf{W}_0^{1,1}(0,1)$ and in fact we are in presence of the Lavrentiev phenomenon, and even of the repulsion property, as one can verify.

4. The repulsion property

The Manià's functional for $\alpha = 9/2$ shows that the Lavrentiev phenomenon does not imply the repulsion property. Nevertheless the implication holds true for any modified Lagrangian $\phi \circ L$, for any function ϕ from $\mathbb{R}$ to $\mathbb{R}$ with super-linear growth, i.e. $\phi(r)/r$ goes to $+\infty$, as r tends to $+\infty$. As particular case, consider $\phi(r) = |r|^\alpha$, with $\alpha > 1$.

More precisely (we recall that the we are in the same framework as the one described at the beginning of Section 3):

Theorem 7. *Let $\bar{u}$ in $\mathbf{W}_\psi^{1,1}(\Omega)$ be a minimizer of $\mathcal{I}$.*

Then,

$$\int_\Omega \phi \circ L(x, u_n, \nabla u_n) \to +\infty,$$

for any sequence $\{\bar{u}_n\}$ in $\mathbf{W}_\psi^{1,\infty}(\Omega)$ such that u_n and ∇u_n converge a.e. respectively to u and ∇u.

Proof. The proof is elementary.

Suppose that there exists a sequence $\{u_n\}$ in $\mathbf{W}_\psi^{1,\infty}(\Omega)$ such that u_n and ∇u_n converge a.e. respectively to u and ∇u and such that

$$\int_\Omega \phi \circ L(x, u_n, \nabla u_n) \to l < +\infty.$$

From the lower bound on the Lagrangian, it follows that $\int_\Omega |\phi \circ L|(x, u_n, \nabla u_n)$ is bounded.

By the De la Vallée–Poussin theorem [1], we have the weak-$*$ compactness in $\mathbf{L}^1(\Omega)$ of $\{L(x, u_n, \nabla u_n)\}$, i.e. there exists a function $\mathcal{L}$ in $\mathbf{L}^1(\Omega)$ that is the weak-$*$ limit of a subsequence of $\{L(x, u_n, \nabla u_n)\}$.

On the other hand, by the Carathéodory assumption on L, we know that $L(x, u_n, \nabla u_n)$ converges a.e. to $L(x, \bar{u}, \nabla \bar{u})$. Hence, $L(x, u_n, \nabla u_n) = \mathcal{L}$ and

$$\mathcal{I}(u_n) \to \mathcal{I}(\bar{u}),$$

that contradicts the presence of the Lavrentiev gap phenomenon. $\square$

Observe that in Theorem 7 it is not required any strong convergence on $\{u_n\}$. The result is true for a.e. convergence.

Corollary 8. *If there exists a function ϕ with super-linear growth, increasing for $r > 0$, such that the functional*

$$\int_{\Omega} \phi^{-1}\big[L(x, u, \nabla u) - \rho\big]\, \mathrm{d}x$$

presents the Lavrentiev phenomenon, then $\mathcal{I}$ manifests the repulsion property, i.e.

$$\mathcal{I}(u_n) = \int_{\Omega} L(x, u_n, \nabla u_n)\, \mathrm{d}x \to +\infty$$

for any sequence $\{\bar{u}_n\}$ in $\mathbf{W}^{1,\infty}_{\psi}(\Omega)$ such that u_n and ∇u_n converge a.e. respectively to u and ∇u.

Proof. By Theorem 7 applied to the functional $\int_{\Omega} \phi^{-1}[L(x, u, \nabla u) - \rho]\, \mathrm{d}x$, we obtain that the energy $\int_{\Omega}[L(x, u, \nabla u) - \rho]\, \mathrm{d}x$ presents the repulsion property. The result therefore follows from the equality:

$$\mathcal{I}(u) = \int_{\Omega} [L(x, u, \nabla u) - \rho]\, \mathrm{d}x + \int_{\Omega} \rho\, \mathrm{d}x. \qquad \square$$

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